How To Make Wavelets

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§1. INTRODUCTION. The French call them *ondelettes*, these new high-tech gadgets in the arsenal of harmonic analysis. Move over, Fourier! Your series and transforms are not the only game in town. Wavelet expansions enjoy a number of good properties not available in other types of expansions. To see this in the simplest context, consider a real-valued function $f(x)$ on the interval $[0,1]$. You can expand it in a Fourier series

$$f(x) = b_0 + \sum_1^{\infty} (b_k \cos 2\pi kx + a_k \sin 2\pi kx) \tag{1.1}$$

or you can expand it in a Haar function series

$$f(x) = c_0 + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} c_{jk} \psi(2^j x - k) \tag{1.2}$$

where $\psi(x)$ is the function defined by

$$\psi(x) = \begin{cases} 1 & \text{if } 0 \le x < \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} \le x < 1 \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

(see Figure 1).

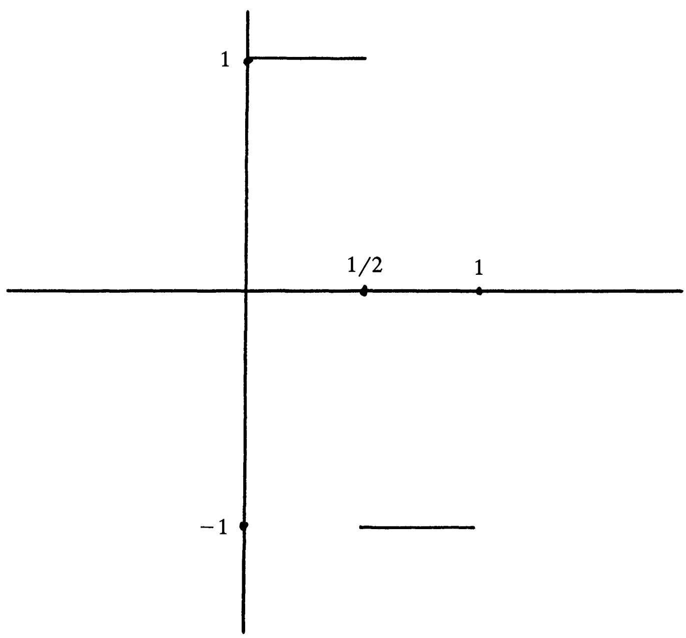

Figure 1. The graph of the generator of the Haar functions.

Both series are examples of expansions in terms of orthogonal functions in $L^2(0,1)$. Thus there are simple formulas for the coefficients. (Exercise: Show that $\{\psi(2^j x - k)\}$ are orthogonal, but not normalized.) But the Fourier series is not well localized in space; if you are interested in the behavior of $f(x)$ on a subinterval $[a,b]$ you need to involve all the Fourier coefficients. On the other hand, the Haar series is very well localized in that to restrict attention to the subinterval $[a,b]$ you need only take the sum in (1.2) over those indices for which the interval $I_{jk} = [2^{-j}k, 2^{-j}(k+1)]$ (the support of $\psi(2^j x - k)$) intersects $[a,b]$. Furthermore, the partial sums of the Haar series (summing $0 \le j \le N$) clearly represents an approximation to f taking into account details on the order of magnitude 2^{-N} or greater. These two properties, *localization in space*, and *scaling*, are the hallmarks of wavelet expansions. In addition, the Haar functions are created out of a single function ψ by dyadic dilations and integer translations. Essentially the same property is shared by all the wavelet bases we will discuss, and may in fact be taken as an approximate definition of a wavelet expansion.

The wavelet expansions we are going to construct can be thought of as generalizations of the Haar series, in which the function ψ is replaced by smoother cousins. Before we can say exactly what properties we want these functions to have, and how we can go about constructing them, it is useful to backtrack and see exactly how the Haar functions arise. It will turn out to be easier if we consider the whole line as the domain of our functions.

§2. THE ROUGH-AND-READY HAAR WAVELETS. We begin with the function φ = characteristic function of the unit interval $[0,1]$. Surely this is one of the simplest functions one can imagine, but it is chosen because it has two important properties:

(i) the translates of φ by integers, $\varphi(x-k)$, $k \in \mathbb{Z}$, form an orthonormal set of functions for $L^2(\mathbb{R})$;

(ii) φ is *self-similar*. If you cut the graph in half then each half can be expanded to recover the whole graph. This property can be expressed algebraically by the *scaling identity*

$$\varphi(x) = \varphi(2x) + \varphi(2x-1). \tag{2.1}$$

We will call φ the *scaling* function. (In the French literature it is sometimes called "le père" and ψ is called "la mère," but this shows a scandalous misunderstanding of human reproduction; in fact the generation of wavelets more closely resembles the reproductive life style of an amoeba.) In fact, the scaling identity essentially determines φ up to a constant multiple (exercise). The significance of the scaling identity is the following: Let V_0 denote the linear span of the functions $\varphi(x-k)$, $k \in \mathbb{Z}$ (or by abuse of notation the closure in $L^2(\mathbb{R})$ of this span, $\sum_{k=-\infty}^{\infty} a_k \varphi(x-k)$ with $\sum |a_k|^2 < \infty$). This is a natural space to consider in view of (i), since the functions $\varphi(x-k)$ form an orthonormal basis for V_0. Of course V_0 is not all of L^2, it is the subspace of piecewise constant functions with jump discontinuities at $\mathbb{Z}$. We can get a larger space by rescaling. Let $(1/2)\mathbb{Z}$ denote the lattice of half-integers $k/2$, $k \in \mathbb{Z}$, and let V_1 denote the subspace of L^2 of piecewise constant functions with jumps at $(1/2)\mathbb{Z}$. It is clear that $f(x) \in V_0$ if and only if $f(2x) \in V_1$, and the functions $2^{1/2}\varphi(2x-k)$ form an orthonormal basis for V_1 (the factor $2^{1/2}$ is thrown in to make the normalization $\|2^{1/2}\varphi(2x-k)\|_2 = 1$ hold). The scaling identity (2.1), or rather its translated version

$$\varphi(x-k) = \varphi(2x-2k) + \varphi(2x-2k-1) \tag{2.1$'$}$$

says exactly $V_0 \subseteq V_1$, since a basis for V_0 is explicitly represented as linear combinations of basis elements of V_1. (Of course the containment $V_0 \subseteq V_1$ is clear from the description of the spaces V_0 and V_1 in terms of locations of jump discontinuities, but in the generalizations to come there will be no such simple description; however, there will be a scaling identity.)

The whole story can now be iterated, both up and down the dyadic scale. The result is an increasing sequence of subspaces V_j for $j \in \mathbb{Z}$, where V_j consists of the piecewise constant L^2 functions with jumps at $2^{-j}\mathbb{Z}$, and the functions $2^{j/2}\varphi(2^j x - k)$ for $k \in \mathbb{Z}$ form an orthonormal basis for V_j. We can pass back and forth among the space V_j by rescaling: $f(x) \in V_j$ if and only if $f(2^{k-j}x) \in V_k$, and the scaling identity (2.1), suitably rescaled, says $V_j \subseteq V_k$ if $j \le k$. The sequence $\{V_j\}$ is an example of what is called a *multiresolution analysis*. There are two other properties of $\{V_j\}$ that are significant, namely

$$\bigcap_{j \in \mathbb{Z}} V_j = \{0\}, \tag{2.2}$$

and

$$\bigcup_{j \in \mathbb{Z}} V_j \text{ is dense in } L^2 \tag{2.3}$$

(exercise).

In view of (2.3) it would seem tempting to try to combine all the orthonormal bases $\{2^{j/2}\varphi(2^j x - k)\}$ of V_j into one orthonormal basis for $L^2(\mathbb{R})$. But look, although $V_j \subseteq V_{j+1}$, the orthonormal basis $\{2^{j/2}\varphi(2^j x - k)\}$ for V_j is not contained in the orthonormal basis $\{2^{(j+1)/2}\varphi(2^{j+1}x - k)\}$ for V_{j+1}. (Indeed, there are distinct elements in the two orthonormal bases that are not orthogonal to each other.) So our first naive attempt to obtain an orthonormal basis for $L^2(\mathbb{R})$ is flawed. Can we fix it up?

Back to the drawing boards! Since $V_0 \subseteq V_1$ and we have an orthonormal basis for V_0 of the form $\{\varphi(x - k)\}$, why don't we try to complete an orthonormal basis of V_1 by adjoining functions of the form $\{\psi(x - k)\}$ for some function ψ? This is the same thing as asking for an orthonormal basis of the desired form for the orthogonal complement of V_0 in V_1, which we denote W_0, so $V_1 = V_0 \oplus W_0$ (Hilbert space direct sum).

The answer is easy: we want to take ψ exactly to be the Haar function generator defined in §1. Note that ψ can be expressed in terms of φ by

$$\psi(x) = \varphi(2x) - \varphi(2x - 1) \tag{2.4}$$

which is very reminiscent of the scaling identity. Exercise: show that $\{\psi(x - k)\}$ forms an orthonormal basis for W_0. But now we can rescale the space W_0, so

$$V_{j+1} = V_j \oplus W_j \tag{2.5}$$

and $\{2^{j/2}\psi(2^j x - k)\}_{k \in \mathbb{Z}}$ is an orthonormal basis for W_j. If we combine conditions (2.2), (2.3) and (2.5) we obtain

$$L^2(\mathbb{R}) = \bigoplus_{j=-\infty}^{\infty} W_j \tag{2.6}$$

and since the spaces W_j are all mutually orthogonal we can now refine our naive attempt and combine all the orthonormal bases for W_j into one grand orthonormal basis $\{2^{j/2}\psi(2^j x - k)\}_{j\in\mathbb{Z},\ k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$. (The only change is that we have replaced the scaling function φ by the wavelet ψ.) This gives the Haar series basis for the whole line. There is a minor variation on this theme that is perhaps more closely related to the Haar expansion on the unit interval: instead of (2.6) we can also write

$$L^2(\mathbb{R}) = V_0 \oplus \left(\bigoplus_{j=0}^{\infty} W_j\right) \tag{2.6$'$}$$

and then combine the basis $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$ for V_0 with the bases $\{2^{j/2}\psi(2^{1/2}x - k)\}_{k\in\mathbb{Z}}$ for W_j with $j \geq 0$, to obtain an orthonormal basis for $L^2(\mathbb{R})$.

§3. MULTIRESOLUTION ANALYSIS. The moral of the story so far is that we first want to build a scaling function φ and associated multiresolution analysis $\cdots \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq \cdots$ before constructing the wavelets.

Definition. A *multiresolution analysis* $\cdots \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq \cdots$ with scaling function φ is an increasing sequence of subspaces of $L^2(\mathbb{R})$ satisfying the following four conditions:

(i) (density) $\bigcup_j V_j$ is dense in $L^2(\mathbb{R})$,

(ii) (separation) $\bigcap_j V_j = \{0\}$,

(iii) (scaling) $f(x) \in V_j \Leftrightarrow f(2^{-j}x) \in V_0$

(iv) (orthonormality) $\{\varphi(x-k)\}_{\gamma\in\mathbb{Z}}$ is an orthonormal basis for V_0.

It follows easily from the definition that $\{2^{j/2}\varphi(2^j x - \gamma)\}_{\gamma\in\mathbb{Z}}$ forms an orthonormal basis for V_j. Since $\varphi \in V_0 \subseteq V_1$ we must have

$$\varphi(x) = \sum_{\gamma\in\mathbb{Z}} a(\gamma)\varphi(2x-\gamma) \tag{3.1}$$

for some coefficients $a(\gamma)$ satisfying

$$\sum_{\gamma\in\mathbb{Z}} |a(\gamma)|^2 = 2 \tag{3.2}$$

and in fact

$$a(\gamma) = 2\int \varphi(x)\overline{\varphi(2x-\gamma)}\,dx. \tag{3.3}$$

Equation (3.1) is the analogue of (2.1), and we will refer to it as the *scaling identity*.

It follows from the definition that the scaling function determines the multiresolution analysis, but not conversely. A more difficult question is how to characterize those functions φ which are scaling functions for a multiresolution analysis. Here we expect the scaling identity to play a crucial role, but before we can say more we need to examine certain algebraic conditions on the coefficients $a(\gamma)$ that follow from the definition.

First, there is a consistency condition that arises from (iv) and (3.1). We know from (iv) that

$$\int \varphi(x-\gamma)\overline{\varphi(x)}\,dx = \delta(\gamma, 0) \tag{3.4}$$

(Kronecker δ). If we use (3.1) to substitute for $\varphi(x-\gamma)$ and $\overline{\varphi(x)}$ in (3.4) we obtain

$$\sum_{\gamma' \in \mathbb{Z}} \sum_{\gamma'' \in \mathbb{Z}} a(\gamma')\overline{a(\gamma'')} \int \varphi(2x - 2\gamma - \gamma')\overline{\varphi(2x - \gamma'')}\, dx$$

$$= 2^{-1} \sum\sum_{\gamma'' = 2\gamma + \gamma'} a(\gamma')\overline{a(\gamma'')} = \delta(\gamma, 0)$$

after the change of variable $x \to 2^{-1}x$ and use of (3.4). We rewrite this as

$$\sum_{\gamma' \in \mathbb{Z}} a(\gamma')\overline{a(2\gamma + \gamma')} = 2\delta(\gamma, 0). \tag{3.5}$$

Note that (3.5) contains (3.2) as a special case.

Another algebraic condition arises if we assume φ is integrable and $\int\varphi(x)\,dx \neq 0$ (if $\int\varphi(x)\,dx = 0$ then the same is true for all functions in all V_j, so we would not expect to have the density condition (i)). Then we integrate (3.1) and make a change of variable to obtain

$$\int \varphi(x)\,dx = \sum_{\gamma \in \mathbb{Z}} a(\gamma) \int \varphi(2x - \gamma)\,dx$$

$$= \sum_{\gamma \in \mathbb{Z}} a(\gamma) 2^{-1} \int \varphi(x)\,dx$$

hence

$$\sum_{\gamma \in \mathbb{Z}} a(\gamma) = 2. \tag{3.6}$$

Now we would like to reverse the procedure. *Step 1* will be to produce solutions $a(\gamma)$ to the algebraic identities (3.5) and (3.6). *Step 2* will be to define the scaling function via the scaling identity (3.1). Notice that (3.1) says that φ is a fixed point of the linear transformation

$$Sf(x) = \sum_{\gamma \in \mathbb{Z}} a(\gamma) f(2x - \gamma) \tag{3.7}$$

so it is reasonable to try to construct φ by iterating S,

$$\varphi = \lim_{n \to \infty} S^n f \tag{3.8}$$

for some reasonable initial function f. In a later section we will discuss another method for solving (3.1). *Step 3* will be to prove that the function φ that solves (3.1) (normalized so $\|\varphi\|_2 = 1$) generates a multiresolution analysis. This is the trickiest step, because there are simple counterexamples to show that it is not always true (try $a(\gamma)$ equal to 1 for $\gamma = 0, 3$, and otherwise $a(\gamma) = 0$, and $\varphi = \chi_{[0,3]}$, which violates (iv)). Nevertheless, many choices of $a(\gamma)$ do yield a multiresolution analysis. The difficult condition to verify is the orthonormality (iv), and we will have to postpone the discussion of when and why this holds to a later section. In Box 1 we will show how to establish the density (i) and separation (ii), given orthonormality and the additional normalization condition

$$\int \varphi(x)\,dx = 1. \tag{3.9}$$

Now we are ready to move on to *Step 4*, which is the construction of the wavelets themselves.

Proofs of Density and Separation

Lemma B1.1. *Let V_0 be any subspace of $L^2(\mathbb{R})$ which is contained in $L^\infty(\mathbb{R})$ and which has the property that*

$$\|f\|_\infty \le c\|f\|_2 \quad \text{for all } f \in V_0. \tag{B1.1}$$

Define V_j by the scaling condition (iii) (no assumption of the sort $V_j \subseteq V_{j+1}$ is necessary). Then (ii) holds.

Proof: The scaling condition and a simple change of variable transforms (B1.1) into

$$\|f\|_\infty \le cm^{j/2}\|f\|_2 \quad \text{for all } f \in V_j. \tag{B1.2}$$

If $f \in \cap V_j$ then (B1.2) holds for all j, and letting $j \to -\infty$ we obtain $\|f\|_\infty = 0$ hence $f = 0$. Q.E.D.

The estimate (B1.1) is easy to obtain in our case. For simplicity assume φ is bounded and has compact support, which will be the case in all our examples. Then by the orthonormality (iv) we have

$$f(x) = \sum_{\gamma \in \mathbb{Z}} \varphi(x-\gamma)\int f(y)\overline{\varphi(y-\gamma)}\,dy = \int K(x,y)f(y)\,dy$$

where $K(x,y) = \Sigma_{\gamma\in\mathbb{Z}}\varphi(x-\gamma)\overline{\varphi(y-\gamma)}$, so

$$|f(x)| \le \left(\int |K(x,y)|^2\,dy\right)^{1/2}\|f\|_2 = \left(\sum_{\gamma\in\mathbb{Z}}|\varphi(x-\gamma)|^2\right)^{1/2}\|f\|_2$$

and $\Sigma_{\gamma\in\mathbb{Z}}|\varphi(x-\gamma)|^2$ is uniformly bounded (of course much weaker conditions on φ, such as rapid decrease will also imply this).

Lemma B1.2. *Assume φ has compact support and satisfies* (3.1) *and* (3.9), *and the orthonormality condition (iv). Then the density condition (i) holds.*

Sketch of Proof: Let $P_j f(x) = 2^j\Sigma_{\gamma\in\mathbb{Z}}\varphi(2^jx-\gamma)\int f(y)\overline{\varphi(2^jy-\gamma)}\,dy$ denote the orthogonal projection onto V_j. We need to show $\lim_{j\to\infty} P_j f = f$ in L^2 for all $f \in L^2$, which is equivalent to $\lim_{j\to\infty}\|P_jf\|_2^2 = \|f\|_2^2$ by the Pythagorean theorem. It suffices to prove this for $f = \chi_A$, A any interval, by a density argument. But $\|P_j\chi_A\|_2^2 = 2^j\Sigma_{\gamma\in\mathbb{Z}}\int_A\varphi(2^jy-\gamma)\,dy^2 = 2^{-j}\Sigma_{\gamma\in\mathbb{Z}}|\int_2^j A\varphi(y-\gamma)\,dy|^2$. For large j, 2^jA will be a large interval, so essentially either $\int_{2^jA}\varphi(y-\gamma)\,dy = 0$ if $\gamma \notin 2^jA$ or $\int_{2^jA}\varphi(y-\gamma)\,dy = 1$ if $\gamma \in 2^jA$ by (3.9) (for γ in a small neighborhood of the boundary of 2^jA this is not quite correct, but in the limit we can ignore this detail). Thus $\|P_j\chi_A\|_2^2 \approx 2^{-j}\#\{\gamma \in 2^jA\} \approx \text{length}(A) = \|\chi_A\|_2^2$ and in the limit this becomes equality. Q.E.D.

Notice that we could essentially reverse the argument to deduce the necessity of the normalization condition (3.9).

§4. THE WAVELETS. We will consider the scaling function φ to be the first element $\varphi = \psi_0$ of a pair of functions ψ_0, ψ_1, with ψ_1 being the wavelet generator. We would like the functions $\{\psi_k(x - \gamma)\}_{\gamma \in \mathbb{Z},\, k=0,1}$ to be an orthonormal basis for V_1. Since the functions $\{\varphi(2x - \gamma)\}_{\gamma \in \mathbb{Z}}$ already form an orthogonal basis for V_1, the functions $\psi_0(x)$ and $\psi_1(x)$ must be linear combinations of $\varphi(2x - \gamma)$, so they must satisfy an identity

$$\psi_k(x) = \sum_{\gamma \in \mathbb{Z}} a_k(\gamma)\varphi(2x - \gamma), \qquad k = 0, 1 \tag{4.1}$$

which generalizes (3.1) (of course $a_0(\gamma) = a(\gamma)$). Notice that for $k = 1$ (4.1) is an explicit formula, there is nothing to solve. But what kind of conditions should we put on the coefficients $a_k(\gamma)$? The same reasoning that led to (3.5) leads to

$$\sum_{\gamma \in \mathbb{Z}} a_j(\gamma')\overline{a_k(2\gamma + \gamma')} = 2\delta(j,k)\delta(\gamma,0). \tag{4.2}$$

On the other hand, the condition $\int\varphi(x)\,dx \neq 0$ is not something we can expect to hold for ψ_1 (think of the example of Haar functions), so conditions (3.6) can only be recopied in our new notation

$$\sum_{\gamma \in \mathbb{Z}} a_0(\gamma) = 2. \tag{4.3}$$

Lemma 4.1. *If* $\{\varphi(x - \gamma)\}_{\gamma \in \mathbb{Z}}$ *is an orthonormal set and if* $a_j(\gamma)$ *satisfy* (4.2) *and* (4.3) *then* $\{\psi_k(x - \gamma)\}_{\gamma \in \mathbb{Z},\, k=0,1}$ *is an orthonormal set.*

Proof: It suffices to show

$$\int \psi_j(x)\overline{\psi_k(x - \gamma)}\,dx = \delta(j,k)\delta(\gamma,0). \tag{4.4}$$

Now

$$\int \psi_j(x)\overline{\psi_k(x - \gamma)}\,dx = \sum_{\gamma' \in \mathbb{Z}} \sum_{\gamma'' \in \mathbb{Z}} a_j(\gamma')\overline{a_k(\gamma'')} \int \varphi(2x - \gamma')\overline{\varphi(2x - 2\gamma - \gamma'')}\,dx.$$

But the integral is $(1/2)\delta(\gamma', 2\gamma - \gamma'')$ by the orthonormality of $\varphi(x - y)$ so (4.4) reduces to (4.2). Q.E.D.

Remark. We have omitted the justification of the interchange of series and integrals, but in most of the examples we will look at the series are actually finite sums.

Thus $\{\psi_k(x - \gamma)\}_{\gamma \in \mathbb{Z},\, k=0,1}$ is an orthonormal set of functions in V_1. Is it a basis? (A kind of pseudo dimension counting argument makes this very plausible.) To show that it is a basis it suffices to represent each function $\varphi(2x - \tilde{\gamma})$ as a linear combination, and we know the coefficients will have to be

$$\int \varphi(2x - \tilde{\gamma})\overline{\psi_k(x - \gamma)}\,dx = \sum \overline{a_k(\gamma')} \int \varphi(2x - \tilde{\gamma})\overline{\varphi(2x - 2\gamma - \gamma')}\,dx$$

$$= \frac{1}{2}\overline{a_k(\tilde{\gamma} - 2\gamma)}.$$

Thus we need to show that

$$\frac{1}{2} \sum_{k=0,1} \sum_{\gamma \in \mathbb{Z}} \overline{a_k(\tilde{\gamma} - 2\gamma)} \psi_k(x - \gamma) \tag{4.5}$$

is equal to $\varphi(2x - \tilde{\gamma})$. But if we substitute (4.1) into (4.5) we obtain

$$\sum_{\gamma \in \mathbb{Z}} \left(\frac{1}{2} \sum_{k=0,1} \sum_{\gamma' \in \mathbb{Z}} \overline{a_k(2\gamma' + \tilde{\gamma})} a_k(2\gamma' + \gamma) \right) \varphi(2x - \gamma)$$

so it suffices to show

$$\sum_{k=0,1} \sum_{\gamma' \in \mathbb{Z}} \overline{a_k(2\gamma' + \tilde{\gamma})} a_k(2\gamma' + \gamma) = 2\delta(\gamma, \tilde{\gamma}), \tag{4.6}$$

for $\tilde{\gamma} = 0$ or 1.

Lemma 4.2. (4.6) *always holds, hence* $\{\psi_k(x - \gamma)\}_{\gamma \in \Gamma,\, k=0,1}$ *is an orthonormal basis for* V_1.

Although this is a purely algebraic statement, we postpone the proof until the next section.

Theorem 4.3. *Suppose* φ *generates a multiresolution analysis and* $a_k(\gamma)$ *satisfy* (4.2) *and* (4.3) *with* ψ_k *defined by* (4.1) *and* $\psi_0 = \varphi$. *Then the functions* $\{2^{j/2}\psi_1(2^j x - \gamma)\}$ *for* $j \in \mathbb{Z}$, $\gamma \in \mathbb{Z}$ *form an orthonormal basis of* $L^2(\mathbb{R})$.

Proof: As before, let W_0 denote the orthogonal complement of V_0 in V_1, $V_1 = V_0 \oplus W_0$. We claim $\{\psi_1(x - \gamma)\}_{\gamma \in \mathbb{Z}}$ is an orthonormal basis for W_0. This follows because we have merely taken the basis for V_1 given by Lemma 4.2 and removed $\{\psi_0(x - \gamma)\}_{\gamma \in \mathbb{Z}}$ which is a basis for V_0. By scaling we obtain

$$V_{j+1} = V_j \oplus W_j$$

and

$$\{2^{j/2}\psi_1(2^j x - \gamma)\}_{\gamma \in \mathbb{Z}}$$

is an orthonormal basis for W_j. But

$$L^2(\mathbb{R}) = \bigoplus_{j \in \mathbb{Z}} W_j$$

by the density condition. Q.E.D.

As a simple variation on the theme, which we leave as an exercise to the reader, the set of functions $\{\varphi(x - \gamma)\}$ for $\gamma \in \mathbb{Z}$ together with $\{2^{j/2}\psi_1(2^j x - \gamma)\}$ for $j \geq 0$, $\gamma \in \mathbb{Z}$ form an orthonormal basis of $L^2(\mathbb{R})$. The advantage of this variant is that we scale only to finer and finer resolutions ($j \to +\infty$) and take care of all the coarser resolutions ($j < 0$) by the single family $\{\varphi(x - \gamma)\}_{\gamma \in \mathbb{Z}}$.

In summary, we have reduced the construction of wavelets to the solution of the algebraic identities (4.2) and (4.3), modulo some technical conditions to ensure the orthonormality condition (iv). *Step 5* will be to actually produce the solutions to (4.2) and (4.3), and *Step 6* will be to establish various properties of the wavelet functions: regularity, decay at infinity, and moment conditions.

The reason we have postponed some of the details in the construction so far is that they require a new technique. So it is now time to open the door and invite Fourier back in.

§5. THE VIEW FROM THE FOURIER TRANSFORM SIDE. Suppose we take the Fourier transform of everything in sight. Because most of our identities have a convolutional structure, we expect a simplification, with multiplicative identities arising in their place. Before doing so, let us return to the orthonormality question, because here the Fourier transform viewpoint gives us an entirely new handle on the problem. Given $\varphi \in L^2$, how can we tell from $\hat{\varphi}$ whether or not $\{\varphi(x-\gamma)\}_{\gamma\in\mathbb{Z}}$ is orthonormal?

It will simplify matters if we adapt the convention (as in **[SW]**) that

$$\hat{\varphi}(x) = \int e^{2\pi i x y}\varphi(y)\,dy \tag{5.1}$$

so that the Fourier inversion formula is just

$$\hat{\hat{\varphi}}(x) = \varphi(-x) \tag{5.2}$$

and the Plancherel formula is

$$\|\varphi\|_2 = \|\hat{\varphi}\|_2 \tag{5.3}$$

(warning: not all the references follow this convention!).

Lemma 5.1. *$\{\varphi(x-\gamma)\}_{\gamma\in\mathbb{Z}}$ is an orthonormal set if and only if*

$$\sum_{\gamma\in\mathbb{Z}} |\hat{\varphi}(\xi+\gamma)|^2 = 1 \quad \text{for all } \xi. \tag{5.4}$$

Proof: By the Plancherel formula, $\{\varphi(x-\gamma)\}_{\gamma\in\mathbb{Z}}$ is orthonormal if and only if

$$\int e^{2\pi i\xi\gamma}|\hat{\varphi}(\xi)|^2\,d\xi = \delta(\gamma,0). \tag{5.5}$$

But the integral over $\mathbb{R}$ can be broken up into an integral over $[0,1]$ and a sum over $\mathbb{Z}$. Since $e^{2\pi i\xi\gamma}$ is periodic we obtain

$$\int_0^1 e^{2\pi i\xi\gamma}\sum_{\gamma\in\mathbb{Z}}|\hat{\gamma}(\xi+\gamma)|^2\,d\xi = \delta(\gamma,0)$$

which means that the function $\Sigma_{\gamma\in\mathbb{Z}}|\hat{\varphi}(\xi+\gamma)|^2$ on $[0,1]$ has as Fourier coefficients $\delta(\gamma,0)$, hence must be the constant function given by (5.4). Q.E.D.

Now the scaling identity (4.1) transcribes easily into the condition

$$\hat{\psi}_k(\xi) = A_k(\tfrac{1}{2}\xi)\hat{\varphi}(\tfrac{1}{2}\xi) \tag{5.6}$$

where

$$A_k(\xi) = \frac{1}{2}\sum_{\gamma\in\mathbb{Z}} a_k(\gamma)e^{2\pi i\gamma\xi} \tag{5.7}$$

(exercise, using the definition of the Fourier transform and a change of variable). Notice that $A_k(\xi)$ is smooth and periodic. Then (4.3) says

$$A_0(0) = 1 \tag{5.8}$$

and (3.9) says

$$\hat{\varphi}(0) = 1. \tag{5.9}$$

By iterating (5.6) for $k = 1$ (remember $\psi_0 = \varphi$) we obtain the infinite product representation

$$\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} A_0(2^{-j}\xi) \tag{5.10}$$

(using (5.8) we can justify the local uniform convergence of the infinite product). Substituting (5.10) back into (5.6) we obtain

$$\hat{\psi}_k(\xi) = A_k(\tfrac{1}{2}\xi) \prod_{j=2}^{\infty} A_0(2^{-j}\xi). \tag{5.11}$$

Thus the functions A_k completely and explicitly determine the wavelets.

The most intricate part of the transcription process is the identity (4.2) that the coefficients $a_k(\gamma)$ must satisfy. What does this tell us about the functions A_k? Rather than deal with this question directly (try it as an exercise, after the fact) we repeat the process which led to (4.2)—namely the consistency of (4.1), alias (5.6), with the orthonormality, alias (5.4). In other words, if $\{\varphi(x - \gamma)\}_{\gamma \in \mathbb{Z}}$ is orthonormal then (5.4) must hold, and if (5.6) defines $\hat{\psi}_k$ then we want the analogue of (5.4), namely

$$\sum_{\gamma \in \mathbb{Z}} \hat{\psi}_k(\xi + \gamma)\overline{\hat{\psi}_j(\xi + \gamma)} = \delta_{jk}. \tag{5.12}$$

Now let $\eta_1 = 0$ and $\eta_2 = 1/2$. These are representations of the cosets of the subgroup $\mathbb{Z}$ in $(1/2)\mathbb{Z}$. Then points of the lattice $\mathbb{Z}$ can be represented uniquely as $2(\gamma + \eta_p)$ as γ varies in $\mathbb{Z}$ and $p = 1, 2$. Then

$$\sum_{\gamma \in \mathbb{Z}} \hat{\psi}_k(\xi + \gamma)\overline{\hat{\psi}_j(\xi + \gamma)} = \sum_{p=1}^{2} \sum_{\gamma \in \mathbb{Z}} \hat{\psi}_k(\xi + 2(\gamma + \eta_p))\overline{\hat{\psi}_j(\xi + 2(\gamma + \eta_p))}$$

by the above parametrization of $\mathbb{Z}$, and if we substitute (5.6) and use the periodicity of A_k we obtain

$$\sum_{p=1}^{2} A_k(\tfrac{1}{2}\xi + \eta_p)\overline{A_j(\tfrac{1}{2}\xi + \eta_p)} \sum_{\gamma \in \mathbb{Z}} \left|\hat{\varphi}(\tfrac{1}{2}\xi + \eta_p + \gamma)\right|^2.$$

The inner sum over $\mathbb{Z}$ yields the constant 1, and so (5.12) yields the consistency condition

$$\sum_{p=1}^{2} A_k(\xi + \eta_p)\overline{A_j(\xi + \eta_p)} = \delta_{jk}. \tag{5.13}$$

This is the Fourier transform equivalent of (4.2). Note that (5.13) implies

$$|A_k(\xi)| \leq 1 \tag{5.14}$$

which implies the boundedness of the Fourier transforms $\hat{\psi}_k$.

We can now easily supply the missing proof of Lemma 4.2. Notice that (5.13) says that for every ξ, the 2×2 matrix $\{A_k(\xi + \eta_p)\}$ is unitary by rows. But this is equivalent to being unitary by columns,

$$\sum_{k=0,1} A_k(\xi + \eta_p)\overline{A_k(\xi + \eta_q)} = \delta_{pq}. \tag{B2.1}$$

Now substituting (5.7) into (B2.1) we obtain

$$\sum_{\gamma \in \mathbb{Z}} \left(\frac{1}{4} \sum_{k=0,1} \sum_{\gamma' \in \mathbb{Z}} a_k(\gamma' + \gamma)\overline{a_k(\gamma')} e^{2\pi i \gamma \eta_p} e^{2\pi i \gamma'(\eta_p - \eta_q)} \right) e^{2\pi i \gamma \xi} = \delta_{pq}.$$

Regarding this as an identity between Fourier series expansions we can equate coefficients to conclude

$$\frac{1}{4} \sum_{k=0,1} \sum_{\gamma' \in \mathbb{Z}} a_k(\gamma' + \gamma)\overline{a_k(\gamma')} e^{2\pi i \gamma \eta_p} e^{2\pi i \gamma'(\eta_p - \eta_q)} = \delta_{pq}\delta(\gamma, 0).$$

Choosing $\eta_p = 0$ and summing over q we obtain (4.6) for $\tilde{\gamma} = 0$ since

$$\sum_{q=1}^{2} e^{-2\pi i \gamma' \eta_q} = \begin{cases} 2 & \text{if } \gamma' \in 2\mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, choosing $\eta_p = 1/2$, multiplying by $e^{2\pi i \eta_q}$ and summing over q we obtain (4.6) for $\tilde{\gamma} = 1$.

The time has come to grasp the bull by the horns and prove the orthonormality of $\{\varphi(x - \gamma)\}_{\gamma \in \mathbb{Z}}$ directly. For this we will need an additional hypothesis.

Theorem 5.2. *Suppose*

$$A_0(\xi) \neq 0 \quad \textit{for } |\xi| \leq \tfrac{1}{4}. \tag{5.15}$$

Then $\{\varphi(x - \gamma)\}_{\gamma \in \mathbb{Z}}$ *is orthonormal.*

Proof: We construct a sequence of functions φ_j such that $\{\varphi_j(x - \gamma)\}_{\gamma \in \mathbb{Z}}$ is orthonormal, and such that $\varphi_j \to \varphi$ in L^2 norm as $j \to \infty$. For φ_0 we simply take $\hat{\varphi}_0(\xi) = \chi_{[-1/2, 1/2]}(\xi)$. Then $\{\varphi_0(\xi - \gamma)\}_{\gamma \in \mathbb{Z}}$ is orthonormal by Lemma 5.1 because (5.4) has exactly one non-zero term.

Inductively define functions φ_j by

$$\hat{\varphi}_j(\xi) = A_0\left(\tfrac{1}{2}\xi\right)\hat{\varphi}_{j-1}\left(\tfrac{1}{2}\xi\right). \tag{5.16}$$

We claim that $\{\varphi_j(x - \gamma)\}_{\gamma \in \mathbb{Z}}$ is again orthonormal. This follows immediately from (5.13) with $j = k = 0$ and Lemma 5.1. It can also be deduced from

$$\varphi_j(x) = \sum_{\gamma \in \mathbb{Z}} a_0(\gamma)\varphi_{j-1}(2x - \gamma) \tag{5.17}$$

which is the non-Fourier transform version of (5.16), and (4.2). Note that

$$\hat{\varphi}_j(\xi) = \left(\prod_{k=1}^{j} A_0(2^{-k}\xi) \right) \chi_{[-2^{j-1}, 2^{j-1}]}(\xi) \tag{5.18}$$

so that $\hat{\varphi}_j \to \hat{\varphi}$ pointwise, by (5.10).

We would like to show $\varphi_j \to \varphi$ in L^2 norm. This will suffice to complete the proof, because the norm limit of orthonormal sets is an orthonormal set. This is the key point of the proof, where the non-vanishing hypothesis must be used. (As an interesting exercise, see how the argument breaks down for the counterexample given in §3.)

By the Plancherel formula it suffices to show $\hat{\varphi}_j \to \hat{\varphi}$ in L^2 norm, and since we have pointwise convergence we would like to use the dominated convergence theorem. Note first that $\hat{\varphi} \in L^2$ by Fatou's theorem, since it is the pointwise limit of $\hat{\varphi}_j$ and $\|\hat{\varphi}_j\|_2 = 1$. Thus we can use a multiple of $\hat{\varphi}$ as a dominator. By comparing (5.18) and (5.10) we see

$$\hat{\varphi}_j(\xi) = \begin{cases} \dfrac{\hat{\varphi}(\xi)}{\hat{\varphi}(2^{-j}\xi)} & \text{if } |\xi| \le 2^{j-1} \\ 0 & \text{otherwise.} \end{cases} \tag{5.19}$$

We claim that $\hat{\varphi}$ is bounded from below on $[-1/2, 1/2]$. The point is that $\hat{\varphi}$ is continuous, and by (5.15) $A_0(2^{-j}\xi) \neq 0$ for $|\xi| \le 1/2$. Thus $\hat{\varphi}$ doesn't vanish on $[-1/2, 1/2]$, so $|\hat{\varphi}_j(\xi)| \le c|\hat{\varphi}(\xi)|$ for $c = (\inf_{[-1/2,1/2]}|\hat{\varphi}|)^{-1}$. Q.E.D.

§6. THE RECIPE. So now we have indicated all the major steps in the construction, but we have left the first to last. We need to find actual solutions to the algebraic identities (5.8), (5.13) and (5.15). There are several different approaches to this problem. We describe one that is due to Ingrid Daubechies **[D1]**.

We look for solutions with only a finite number of $a_k(\gamma)$ different from zero, which means $A_k(\xi)$ are trigonometric polynomials. This implies that the scaling function φ and wavelet ψ_1 have compact support. This can be seen most easily from the iteration procedure (3.7) and (3.8). Say $a(\gamma) = 0$ unless $\gamma \in [0, N]$; then if f has support in $[0, N]$, so does Sf.

We concentrate first on finding the function A_0, which must satisfy three conditions:

$$A_0(0) = 1 \tag{6.1}$$

$$|A_0(\xi)|^2 + |A_0(\xi + \tfrac{1}{2})|^2 = 1 \tag{6.2}$$

$$A_0(\xi) \neq 0 \quad \text{for} \quad |\xi| \le \tfrac{1}{4} \tag{6.3}$$

(here (6.1) is (5.8), (6.2) is (5.13) for $j = k = 0$ and (6.3) is (5.15)). And, of course, A_0 must be of the form

$$A_0(\xi) = \frac{1}{2} \sum_{\gamma \in \mathbb{Z}} a_0(\gamma) e^{2\pi i \gamma \xi} \quad \text{(finite sum)}. \tag{6.4}$$

Note that $|A_0(\xi)|^2$ is then of the same form.

Now we already know one solution, namely

$$A_0(\xi) = \tfrac{1}{2}(1 + e^{2\pi i \xi}) = e^{\pi i \xi} \cos \pi \xi$$

which yields the Haar wavelets. This was deemed unsatisfactory because the wavelets are not continuous. One way to create continuity and even differentiability is to take convolution powers, or on the Fourier transform side to take ordinary powers. Thus we are tempted to try $A_0(\xi) = (e^{\pi i \xi} \cos \pi \xi)^N$ for some large N. Unfortunately (6.2) no longer holds, but we can fix this up. Note that $\cos \pi(\xi + 1/2) = -\sin \pi \xi$, so that is why $|\cos \pi \xi|^2 + |\cos \pi(\xi + 1/2)|^2 = 1$.

Now take the identity $\cos^2 \pi\xi + \sin^2 \pi\xi = 1$ and raise it to an odd power, say

$$\begin{aligned}1 &= (\cos^2 \pi\xi + \sin^2 \pi\xi)^5 \\ &= \cos^{10} \pi\xi + 5\cos^8 \pi\xi \sin^2 \pi\xi + 10\cos^6 \pi\xi \sin^4 \pi\xi \\ &\quad + 10\cos^4 \pi\xi \sin^6 \pi\xi + 5\cos^2 \pi\xi \sin^8 \pi\xi + \sin^{10} \pi\xi.\end{aligned}$$

Take the first half of the terms for $|A_0|^2$,

$$|A_0(\xi)|^2 = \cos^{10} \pi\xi + 5\cos^8 \pi\xi \sin^2 \pi\xi + 10\cos^6 \pi\xi \sin^4 \pi\xi. \tag{6.5}$$

Replacing ξ by $\xi + 1/2$ turns these into the second half of the terms, so (6.2) is automatic, and (6.1) and (6.3) are easy. This gives a recipe for producing $|A_0|^2$, and it remains to take a square root of the form (6.4). We would also like to take the coefficients $a_0(\gamma)$ in (6.4) to be real, for that will yield a real-valued scaling function (and in the end real-valued wavelets as well). There is a general theorem of F. Riesz that asserts that this is possible, but in this case it is easy enough to accomplish by trial and error. Since

$$\begin{aligned}|A_0(\xi)|^2 &= \cos^6 \pi\xi(\cos^4 \pi\xi + 5\cos^2 \pi\xi \sin^2 \pi\xi + 10\sin^4 \pi\xi) \\ &= \cos^6 \pi\xi\Big((\cos^2 \pi\xi - \sqrt{10}\sin^2 \pi\xi)^2 + (5 + 2\sqrt{10})\cos^2 \pi\xi \sin^2 \pi\xi\Big)\end{aligned}$$

we can take

$$\begin{aligned}A_0(\xi) &= (e^{\pi i\xi}\cos \pi\xi)^3\Big(\cos^2 \pi\xi - \sqrt{10}\sin^2 \pi\xi + i\sqrt{5 + 2\sqrt{10}}\cos \pi\xi \sin \pi\xi\Big) \\ &= \frac{1}{8}(e^{2\pi i\xi} + 1)^3\bigg(\frac{1 - \sqrt{10}}{2} + \frac{1 + \sqrt{10}}{4}(e^{2\pi ix} + e^{-2\pi ix}) \\ &\qquad + \frac{1}{4}\sqrt{5 + 2\sqrt{10}}\,(e^{2\pi ix} - e^{-2\pi ix})\bigg)\end{aligned} \tag{6.6}$$

which is clearly of the form (6.4) with $a_0(\gamma)$ real and $a_0(\gamma) \neq 0$ only if $-1 \leq \gamma \leq 4$.

To complete the story we need to find $A_1(\xi)$, also of the form (6.4), which satisfies

$$|A_1(\xi)|^2 + |A_1(\xi + \tfrac{1}{2})|^2 = 1 \tag{6.7}$$

and

$$A_0(\xi)\overline{A_1(\xi)} + A_0(\xi + \tfrac{1}{2})\overline{A_1(\xi + \tfrac{1}{2})} = 0 \tag{6.8}$$

(these are the remaining conditions of (5.13)). Fortunately, this can be accomplished just by taking

$$A_1(\xi) = e^{2\pi i\xi}\overline{A_0(\xi + \tfrac{1}{2})} \tag{6.9}$$

which amounts to setting

$$a_1(\gamma) = (-1)^{\gamma+1}\overline{a_0(1 - \gamma)}. \tag{6.10}$$

Then (6.7) and (6.8) follow directly from (6.2) and the periodicity of A_0. Note also that $a_1(\gamma)$ are real valued if $a_0(\gamma)$ are.

The Fourier transform of ψ_1 is given by (5.11), which now reads

$$\hat{\psi}_1(\xi) = A_1(\tfrac{1}{2}\xi)\prod_{j=2}^{\infty} A_0(2^{-j}\xi) \tag{6.11}$$

with A_0 given by (6.6) and A_1 by (6.9). If we want to obtain the wavelet ψ_1 itself rather than its Fourier transform we first find $\psi_0 = \varphi$ by iterating the mapping

$$Sf(x) = \sum_{\gamma} a_0(\gamma) f(2x - \gamma) \tag{6.12}$$

starting with any reasonable f satisfying $\int f(x)\,dx = 1$, and then setting

$$\psi_1(x) = \sum_{\gamma} a_1(\gamma) \varphi(2x - \gamma). \tag{6.13}$$

See FIGURES 2 and 3.

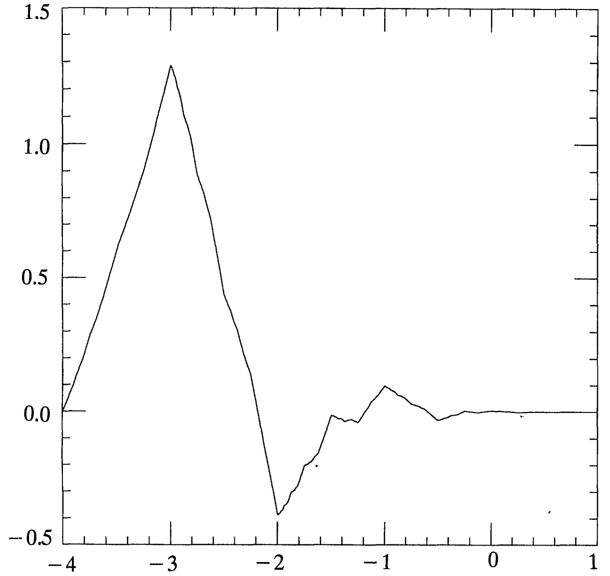

Figure 2. The graph of the scaling function φ, courtesy of David Aronstein.

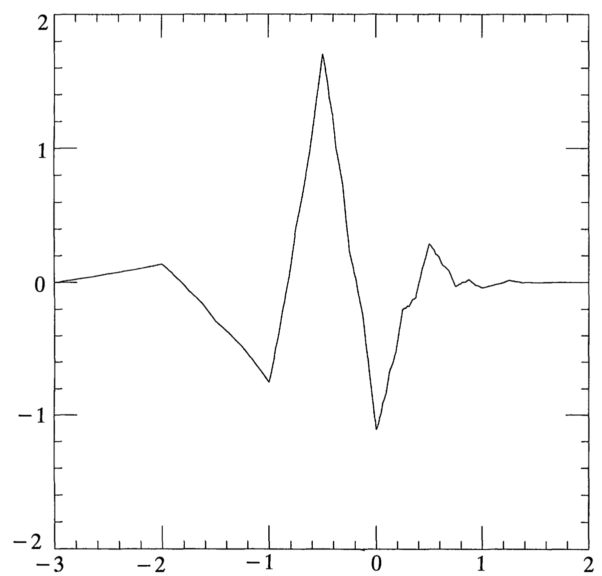

Figure 3. The graph of the wavelet generator ψ_1, courtesy of David Aronstein.

There is an alternative approach to constructing the scaling function that yields a different wavelet basis. It has the advantage of requiring less algebra, but the disadvantage of producing wavelets that are not compactly supported. Start with the Haar basis scaling function $\chi_{[0,1]}$, whose Fourier transform is $e^{\pi i \xi}(\sin \pi\xi/\pi\xi)$, and take the N-fold convolution product

$$g = \chi_{[0,1]} * \chi_{[0,1]} * \cdots * \chi_{[0,1]} \qquad (N \text{ factors})$$

so that

$$\hat{g}(\xi) = \left(e^{\pi i \xi}\frac{\sin \pi\xi}{\pi\xi}\right)^N.$$

It is easy to see that $g \in C^{N-1}$, but of course we have destroyed the orthonormality of translates by $\mathbb{Z}$ that $\chi_{[0,1]}$ had. Too bad, but this is easily fixed. Write

$$h(\xi) = \left(\sum_{k\in\mathbb{Z}} |\hat{g}(\xi + k)|^2\right)^{1/2}$$

and observe that h is periodic and

$$0 < c_1 \le h(\xi) \le c_2 < \infty.$$

Then we have only to take

$$\hat{\varphi}(\xi) = \hat{g}(\xi)/h(\xi)$$

and (5.4) is automatic, so we have the orthonormality of $\{\varphi(x-\gamma)\}_{\gamma\in\mathbb{Z}}$. Notice that $\hat{g}(0) = 1$ and $\hat{g}(\gamma) = 0$ for $\gamma \neq 0$ so $\hat{\varphi}(0) = 1$ as required. And it is not difficult to show that $\varphi \in C^{N-1}$.

What about the scaling identity? Well, it certainly holds for g, namely

$$\hat{g}(\xi) = B(\xi/2)\hat{g}(\xi/2)$$

where

$$B(\xi) = \left(e^{\pi i \xi}\cos \pi\xi\right)^N$$

has the required form (6.4). It then follows that

$$\hat{\varphi}(\xi) = A_0(\xi/2)\hat{\varphi}(\xi/2)$$

where

$$A_0(\xi) = B(\xi)h(\xi)/h(2\xi).$$

Now A_0 is periodic, so it must have the form (6.4), but the sum is no longer finite. This is where we lose the compact support of φ. On the other hand A_0 is clearly smooth, so the Fourier coefficients in (6.4) must be rapidly decreasing, which implies that φ is rapidly decreasing.

The construction of $A_1(\xi)$ and the wavelet Fourier transform $\hat{\psi}_1(\xi)$ then proceeds via (6.9) and (6.11) as before.

§7. SMOOTHNESS OF WAVELETS. How smooth are our wavelets? Since we understand them best on the Fourier transform side, we will use the principle that decay at infinity of $\hat{\varphi}$ implies smoothness of φ (we will establish smoothness of the scaling function and pass it on to the wavelets via (6.13)). For example, it is easy to show

$$|\hat{\varphi}(\xi)| \leq c(1 + |\xi|)^{-N-1-\varepsilon} \tag{7.1}$$

implies $\varphi \in C^N$. So how do we establish (7.1)?

We have the infinite product representation (5.10) which says

$$\hat{\varphi}(\xi) = \prod_{k=1}^{\infty} A_0(2^{-k}\xi) \tag{7.2}$$

and A_0 is periodic. Since each factor does not decay at infinity, why should the product? This is a mystery, which is best solved by looking at the simplest case, $A_0(\xi) = \cos \pi\xi$. Then

$$\prod_{k=1}^{\infty} \cos 2^{-k}\pi\xi = \frac{\sin \pi\xi}{\pi\xi} \tag{7.3}$$

does decay at the rate $O(|\xi|^{-1})$. (Formula (7.3) was proved by Euler, but special cases were known by Francois Viète in the late 1500's. You can prove it by considering the Fourier transform of $\chi_{[-1/2,1/2]}$ and its scaling properties.)

Clearly, for most choices of ξ, the values of $\cos 2^{-k}\pi\xi$ will occasionally become small, and that makes the product (7.3) small. You might try to get around this by taking $\xi = 2^N$ for large N. Thus $\cos 2^{-k}\pi\xi = \pm 1$ for $k = 1, \ldots, N$, so there is no decay, but then $\cos 2^{-N-1}\pi\xi = 0$ wipes you out. You can try to quantify this line of reasoning, but there is no great payoff in showing, for example, that $\sin \pi\xi/\pi\xi = O(|\xi|^{-2/3})$, so we will take (7.3) as our starting point.

The expression (6.6) for A_0, or any of its more complicated cousins, contains $\cos\pi\xi$ as a factor, many times. Thus $\hat{\varphi}(\xi)$ contains $\sin \pi\xi/\xi$ as a factor many times, hence we expect decay. Unfortunately, the other factor grows. It is easier to work with $|A_0|^2$ given by (6.5), if we remember to take the square root at the end. We have, for the special case considered,

$$|A_0(\xi)|^2 = (\cos \pi\xi)^6(\cos^4 \pi\xi + 5\cos^2 \pi\xi \sin^2 \pi\xi + 10\sin^4 \pi\xi).$$

The first factor produces decay $O(|\xi|^{-6})$. The second factor can be written $1 + 3\sin^2 \pi\xi + 6\sin^4 \pi\xi$ so it clearly has a maximum value 10 at $\xi = 1/2$. We can obtain a crude estimate for the growth rate produced by the second factor by the following reasoning: if $|\xi| \approx 2^N$ then there will be about N factors where $2^{-k}|\xi|$ is large, so an upper bound for the product is a constant times 10^N. But $10^N \approx |\xi|^\alpha$ for $\alpha = \log 10/\log 2 \approx 3.32$. So the growth rate is at most $O(|\xi|^{3.32})$ so the combination gives $O(|\xi|^{-2.68})$ for $|\hat{\varphi}(\xi)|^2$ hence $O(|\xi|^{-1.34})$ for $\hat{\varphi}(\xi)$.

This is a disappointing estimate. According to (7.1) it suffices only to show that φ is continuous. It can be improved, but not by a lot. To see why, consider $\xi = 2^N/3$. Then for each of the N factors $2^{-k}\xi = 2^{N-k}/3$, $1 \leq k \leq N$, we have $1 + 3\sin^2 2^{N-k}\pi/3 + 6\sin^4 2^{N-k}\pi/3 = 1 + 3 \cdot (\sqrt{3}/2)^2 + 6(\sqrt{3}/2)^4 = 6.625$ so a lower bound for α is $\log 6.625/\log 2$ which yields $O(|\xi|^{-1.636})$ as the optimal improvement.

If we consider the family of wavelets constructed as outlined in §6, we will have $|A_0(\xi)|^2$ written as the product of higher and higher powers of $\cos \pi\xi$ by more and more complicated second factors. Thus we have faster decay times faster growth in $\hat{\varphi}(\xi)$. Which wins? Well, it is a close race! It turns out that the decay wins, but the

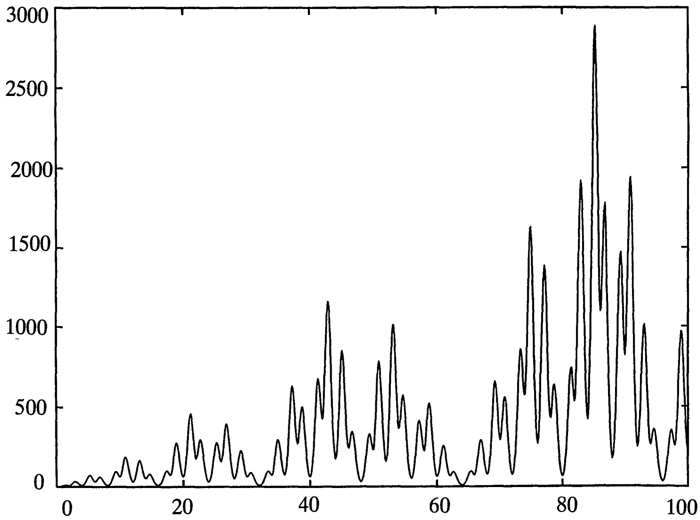

Figure 4. The graph of $\hat{\varphi}$, after factoring out a power of $\sin \pi x/\pi x$, courtesy of Prem Janardhan and David Rosenblum.

crude method of estimating the growth used above is not good enough to show this. The final result ([**D1**], [**C2**]) is that to create wavelets of class C^N we need to carry out the construction starting with $(\cos^2 \pi\xi + \sin^2 \pi\xi)^M = 1$ for M on the order of $5(N+1)$. This means that there is a rather high price to pay in terms of complexity (the algebra required to pass from $|A_0|^2$ to A_0, for example) in order to gain a moderate amount of smoothness. (More recently, better techniques have been found to estimate the smoothness directly, without involving the Fourier transform [**DL**].) FIGURE 4 shows the graph of $\hat{\varphi}(\xi)$. See [**JRS**] for a discussion of the surprising self-similarity properties of this function.

In addition to smoothness, another important property of wavelets is the vanishing moment conditions

$$\int_{-\infty}^{\infty} x^k \psi_1(x)\, dx = 0, \quad k = 0, 1, \ldots, N \tag{7.4}$$

which are equivalent to the vanishing of the Fourier transform to high order at the origin,

$$\left(\tfrac{d}{d\xi}\right)^k \hat{\psi}_1(0) = 0, \quad k = 0, 1, \ldots, N. \tag{7.5}$$

In contrast to smoothness, however, it is only the wavelet, not the scaling function, which enjoys this property. The significance of this condition is that it implies a weak form of localization in the frequency (Fourier transform) variable, since the Fourier transform of $\psi_1(2^j x - k)$ is mainly concentrated around values of $|\xi|$ on the order of 2^j. (There is yet another family of wavelets in which the Fourier transform is actually supported in an annular region $c_1 2^j \le |\xi| \le c_2 2^j$. See [**M**] for a description of these "Littlewood-Paley" type wavelets.) For our wavelets the verification of (7.5) is easy. From (6.11) we see that $\hat{\psi}_1$ has a factor $A_1((1/2)\xi)$, and from (6.9) we see that A_1 at $\xi = 0$ has the same order zero as A_0 at $\xi = 1/2$. But A_0 has a factor of $\cos \pi\xi$ to a power, hence vanishes at $\xi = 1/2$ to order 3 in our particular example, and to order M if we start with $(\cos^2 \pi x + \sin^2 \pi x)^M = 1$ in our construction. Note that in general conditions (6.1) and (6.2) imply that

$A_0(1/2) = 0$, and the flatter we make A_0 near $\xi = 0$, the more it vanishes near $\xi = 1/2$.

§8. CONCLUDING REMARKS. Why not try to create your own designer wavelets by programming the recipe given in §6, and taking the square root of $|A_0(\xi)|^2$ in a different way? For a more detailed discussion of the Riesz Lemma for doing this see **[D1]**.

For further information about wavelets, including historic accounts and attribution of results, see the books **[M]**, **[BF]**, **[BC]** or the expository lectures **[D2]** and **[FJW]**. The term "wavelet" is also used to describe expansions in terms of functions which are not orthogonal. These wavelets have a simpler algebraic description, which is useful for some applications. An expanded version of this article, including a discussion of wavelet bases in several variables, will appear in **[BF]**. None of the theorems or proofs presented here are original; I have only tried to organize the material in a way that is easy to digest.

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